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1990 J. Phys. A: Math. Gen. 23 3483

(http://iopscience.iop.org/0305-4470/23/15/021)

J. Phys. A: Math. Gen. **23** (1990) 3483-3496. Printed in the UK

An orthonormal basis of solenoidal vector functions vanishing on a cylinder

Martin Macháček

Institute of Astronomy, Czechoslovak Academy of Sciences, 251 65 Ondřejov, Czechoslovakia

Received 22 November 1989

Abstract. We derive expressions for vector fields $w_{knl}(r, \varphi, z)$ defined inside a cylinder and satisfying the following conditions: (a) $\nabla \cdot w_{knl} = 0$; (b) $w_{knl}(R, \varphi, z) = 0$; (c) $w_{knl}(r, \varphi, z + Z) = w_{knl}(r, \varphi, z)$, where R and Z are constants; (d) $\Delta w_{knl} + \lambda^2_{knl} w_{knl} = \nabla p_{knl}$ for some scalar p_{knl}; (e) $-\mathrm{i}\partial_z w_{knl} = k w_{knl}$; (f) $j w_{knl} = n w_{knl}$, where $j = s - i\partial_\varphi$ is the generator of rotations about the axis ('spin + orbital momentum' operator); (g) $\int |w_{knl}|^2 \, \mathrm{d}V = 1$, where the integration is over the cylinder of length Z with radius R. These functions form an orthonormal basis complete on the space L_2 of vector functions satisfying (a), (b) and (c). The basis can be used for studying, e.g., the stability of laminar pipe flow, or pipe turbulence.

1. Introduction

Despite its apparent simplicity, the question about stability of laminar pipe flow is still not satisfactorily solved. The reason is clear. Owing to the presence of the pressure term $-\nabla p$, the linearization of the Navier-Stokes equations

$$\partial \boldsymbol{v}/\partial t = -(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nu \Delta \boldsymbol{v} + f - \nabla p \equiv \boldsymbol{Q}(\boldsymbol{v}) - \nabla p \tag{1.1}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{1.2}$$

does not lead to the eigenvalue problem for an explicitly given linear operator. The same pressure term is a nuisance in many other hydrodynamical problems. In an attempt to get rid of it, we now examine its mathematical meaning.

The incompressibility condition (1.2) together with the boundary conditions define a set of functions, $\mathscr{F}$, to which the solution $\boldsymbol{v}(x, t)$ of (1.1) must belong at any time. Often the boundary conditions are expressed by a linear homogeneous equation, such as

$$\boldsymbol{v} = 0 \qquad \text{on the boundary} \tag{1.3}$$

and then $\mathscr{F}$ is a linear functional space. If $\boldsymbol{v}(., t) \in \mathscr{F}$ for any t, then so must its time derivative and the right-hand side of (1.1). However, the apparent contributions to $\partial \boldsymbol{v}/\partial t$, collected in the operator $Q(v)$, do not in general belong to $\mathscr{F}$: the inertia term $-(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ satisfies (1.3) but not (1.2), the viscosity term $\nu \Delta \boldsymbol{v}$ satisfies (1.2) but not (1.3), and the external volume force f may or may not satisfy anything. Therefore, to keep $\boldsymbol{v}$ in $\mathscr{F}$, we must subtract something from $\boldsymbol{Q}(\boldsymbol{v})$; we subtract the gradient of a scalar p to get the orthogonal projection of $\boldsymbol{Q}(\boldsymbol{v})$ on $\mathscr{F}$, as in figure 1. (With the functional scalar product defined by

$$(u, v) = \int \boldsymbol{u}^* \cdot \boldsymbol{v} \, \mathrm{d}V$$

gradients are orthogonal to the space $\mathscr{F}$.)

There are, however, easier ways of projecting a vector onto a space than finding the orthogonal component of $\boldsymbol{Q}(\boldsymbol{v})$ and then subtracting it. The most obvious way is to have an orthonormal basis $\{\boldsymbol{w}_l\}$ in $\mathscr{F}$, expand $\boldsymbol{Q}(\boldsymbol{v})$ in this basis, and leave the rest, $\boldsymbol{Q}(\boldsymbol{v}) - \Sigma(\boldsymbol{w}_l, \boldsymbol{Q}(\boldsymbol{v}))\boldsymbol{w}_l$, to take care of itself instead of cancelling it explicitly with $-\nabla \boldsymbol{p}$. In this paper we construct such a basis for the case when the boundary conditions used in the definition of $\mathscr{F}$ are those specific to the pipe flow: vanishing on a cylinder, and periodicity in the axial direction. (The union of all such bases with all positive periods in the axial direction is a basis for the infinitely long pipe flow.) Using this basis, the problem of the stability of the laminar pipe flow is considerably simplified and becomes tractable even in the non-axisymmetric case. The basis is also very convenient for treating turbulent pipe flows. As solenoidal vector fields are important in other areas of physics, e.g. in field theory, some applications of this basis may be found even outside hydrodynamics.

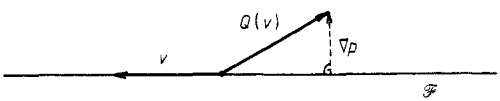

Figure 1. Symbolic representation of the terms of (1.1) as vectors in a functional space.

2. Non-normalized eigenfunctions

The basis we seek consists of vector functions $\boldsymbol{v}_l(r, \varphi, z)$ satisfying the relations

$$\nabla \cdot \boldsymbol{w}_l = 0 \tag{2.1}$$

$$\boldsymbol{w}_l(R, \varphi, z) = 0 \tag{2.2}$$

$$\boldsymbol{w}_l(r, \varphi, z+Z) = \boldsymbol{w}_l(r, \varphi, z) \tag{2.3}$$

which define the space $\mathscr{F}$; here r, φ and z are the cylindrical coordinates, R is the radius and Z the length of the pipe. An easy way of constructing orthonormal bases is to take eigenfunctions of suitable self-adjoint operators. We shall use three mutually commuting operators: the projection of the Laplacian on $\mathscr{F}$, the generator of rotations about the axis and the generator of shifts along the axis.

As mentioned in section 1, the Laplacian is not defined on $\mathscr{F}$: in general, $\Delta \boldsymbol{w} \notin \mathscr{F}$ even if $\boldsymbol{w} \in \mathscr{F}$, since $\Delta \boldsymbol{w}$ need not satisfy the boundary condition (2.2). However, adding the gradient of a suitable scalar p, we get into $\mathscr{F}$ again (cf figure 1). Therefore the operator $\tilde{\Delta}$, the projection of Δ on $\mathscr{F}$, is defined, for $\boldsymbol{u}$, $\boldsymbol{v} \in \mathscr{F}$, by

$$\tilde{\Delta}\boldsymbol{u} = \boldsymbol{v} \qquad \text{if } \Delta\boldsymbol{u} + \nabla p = \boldsymbol{v} \qquad \text{for some } p$$

and its eigenfunctions satisfy

$$\Delta \boldsymbol{w}_l + \nabla p = -\lambda^2 \boldsymbol{w}_l. \tag{2.4}$$

Ladyzhenskaya (1969) proves that $\tilde{\Delta}$ is self-adjoint, and by partial integration we can easily see that it is negative definite; therefore we write its eigenvalues as $-\lambda^2$.

The generator of shifts along the axis is, up to a constant factor, equal to $-\mathrm{i}\partial_z$. The corresponding operator for rotations about the axis is not, however, $L=-\mathrm{i}\partial_\varphi$, but $J=L+S$, where the spin operator S is defined by its action on the unit vector fields $\boldsymbol{e}_+$, $\boldsymbol{e}_-$ and $\boldsymbol{e}_z$:

$$\boldsymbol{e}_\pm = 2^{-1/2}(\boldsymbol{e}_r \pm \mathrm{i}\boldsymbol{e}_\varphi) \tag{2.5}$$

as

$$S\boldsymbol{e}_\pm = \pm\boldsymbol{e}_\pm .$$

(Here $\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_z$ are the orthogonal unit vectors of the cylindrical coordinate system.) From the obvious relations

$$\partial_\varphi \boldsymbol{e}_r = \boldsymbol{e}_\varphi \qquad \partial_\varphi \boldsymbol{e}_\varphi = -\boldsymbol{e}_r$$

we obtain

$$L\boldsymbol{e}_\pm = -\mathrm{i}\partial_\varphi \boldsymbol{e}_\pm = \mp \boldsymbol{e}_\pm \tag{2.6}$$

$$J\boldsymbol{e}_\pm = (L+S)\boldsymbol{e}_\pm = 0 \tag{2.7}$$

which expresses the rotation invariance of the vector fields $\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$ and their linear combinations $\boldsymbol{e}_+$, $\boldsymbol{e}_-$. (In contrast, the 'orbital' operator $L=-\mathrm{i}\partial_\varphi$ gives $L\boldsymbol{e}_x = L\boldsymbol{e}_y = 0$, though the constant unit vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are by no means rotation invariant.) Therefore the eigenfunctions $\boldsymbol{w}_I$ must satisfy

$$-\mathrm{i}\partial_z \boldsymbol{w}_I = k\boldsymbol{w}_I \tag{2.8}$$

$$J\boldsymbol{w}_I = n\boldsymbol{w}_I . \tag{2.9}$$

We shall express them in terms of the unit vectors $\boldsymbol{e}_+$, $\boldsymbol{e}_-$ and $\boldsymbol{e}_z$:

$$\boldsymbol{w} = \boldsymbol{e}_+ w_- + \boldsymbol{e}_- w_+ + \boldsymbol{e}_z w_z = (\boldsymbol{e}_+, \boldsymbol{e}_-, \boldsymbol{e}_z)\begin{pmatrix} w_- \\ w_+ \\ w_z \end{pmatrix} \tag{2.10}$$

(for simplicity we drop the subscript I where it is not necessary). From (2.7) it follows that

$$J\boldsymbol{w} = (\boldsymbol{e}_+, \boldsymbol{e}_-, \boldsymbol{e}_z)(-\mathrm{i}\partial_\varphi)\begin{pmatrix} w_- \\ w_+ \\ w_z \end{pmatrix}$$

therefore (2.8) and (2.9) can be rewritten as

$$-\mathrm{i}\partial_z w_\alpha = k w_\alpha \tag{2.11}$$

$$\alpha = +, -, z.$$

$$-\mathrm{i}\partial_\varphi w_\alpha = n w_\alpha \tag{2.12}$$

The solution is obviously

$$w_\alpha(r, \varphi, z) = W_\alpha(r)\, \mathrm{e}^{\mathrm{i}kz}\, \mathrm{e}^{\mathrm{i}n\varphi}. \tag{2.13}$$

As $\boldsymbol{w}$ is single-valued, n must be integer; the boundary condition (2.3) implies that k must be an integer multiple of $2\pi/Z$.

The pressure p in the eigenvalue equation (2.4) must have the same sinusoidal dependence on φ and z as w_α:

$$p(r, \varphi, z) = P(r)\, e^{ikz}\, e^{in\varphi}. \tag{2.14}$$

Applying the divergence operator on (2.4) and using (2.1), we find that p is a harmonic function:

$$\Delta p \equiv \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2 + \partial_z^2\right)p = 0. \tag{2.15}$$

We insert (2.14) into (2.15):

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2} - k^2\right)P(r) = 0. \tag{2.16}$$

The solution of this equation which is finite at the origin is

$$P(r) = \begin{cases} \text{constant} \times r^{|n|} & \text{if } k = 0 \qquad (2.17) \\ \text{constant} \times I_n(kr) & \text{if } k \neq 0 \qquad (2.18) \end{cases}$$

where I_n are the modified Bessel functions.

As is natural in the cylindrical geometry, we shall frequently deal with the Bessel and modified Bessel functions; to avoid involved formulae, we introduce the operators

$$D_n^\pm = \partial_r \mp \frac{n}{r} \tag{2.19}$$

and recall that the following relations hold (see, e.g., Abramowitz and Stegun (1965)):

$$D_n^\pm I_n(kr) = kI_{n\pm1}(kr) \tag{2.20}$$

$$D_n^\pm J_n(ar) = \mp aJ_{n\pm1}(ar) \tag{2.21}$$

$$D_{n-1}^+ D_n^- = D_{n+1}^- D_n^+ = \partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2} \tag{2.22}$$

$$D_{n-1}^+ D_n^- J_n(ar) = -a^2 J_n(ar) \tag{2.23}$$

$$D_{n-1}^+ D_n^- I_n(kr) = k^2 I_n(kr). \tag{2.24}$$

(Equations (2.23) and (2.24) are the Bessel equations for J_n and I_n.) Relations analogous to (2.20) but involving the solution (2.17) instead of (2.18) are

$$D_n^\pm r^{|n|} = 2|n|\, Y(\mp n) r^{|n|-1} \tag{2.25}$$

where Y is the Heaviside function: $Y(n) = 1$ if $n \geqslant 0$, $Y(n) = 0$ if $n < 0$.

Now we express (2.4) in the coordinate system (e_+, e_-, e_z). The pressure (2.14) enters (2.4) through its gradient:

$$\begin{aligned} \nabla p &= \left(e_r\partial_r + e_\varphi \frac{\partial_\varphi}{r} + e_z\partial_z\right)p \\ &= [2^{-1/2}(e_+D_n^- + e_-D_n^+) + e_z\, ik]P(r)\, e^{in\varphi}\, e^{ikz}. \end{aligned} \tag{2.26}$$

The Laplacian operator can be expressed in terms of three auxiliary operators, $L = -i\partial_\varphi$ and

$$D^\pm = e^{\pm i\varphi}\left(\partial_r \mp \frac{L}{r}\right) \tag{2.27}$$

as

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r - \frac{L^2}{r^2} + \partial_z^2 = D^+D^- + \partial_z^2 = D^-D^+ + \partial_z^2. \tag{2.28}$$

If w_α is a scalar function of the form (2.13), then

$$D^+D^- w_\alpha = D^+_{n-1}D^-_n w_\alpha \tag{2.29}$$

$$D^-D^+ w_\alpha = D^-_{n+1}D^+_n w_\alpha. \tag{2.30}$$

However, when applying these operators to a vector function, we must take into account the commutation relation

$$L(e_\pm w_\alpha) = e_\pm(L \mp 1) w_\alpha \tag{2.31}$$

following from (2.6). Therefore

$$\begin{aligned} D^-D^+(e_+ w_-) &= e_+(D^-_n D^+_{n-1} w_-) \\ D^+D^-(e_- w_+) &= e_-(D^+_n D^-_{n+1} w_+) \end{aligned} \tag{2.32}$$

and the Laplacian acts in the following way:

$$\begin{aligned} &\Delta(e_+ w_+ + e_- w_+ + e_z w_z) \\ &\quad = e_+(D^-_n D^+_{n-1} - k^2) w_- + e_-(D^+_n D^-_{n+1} - k^2) w_+ + e_z(D^+_{n-1}D^-_n - k^2) w_z. \end{aligned} \tag{2.33}$$

In solving (2.4) we distinguish four principal cases according to the value of k and n.

Case 1. $k > 0$, $n > 0$. According to (2.26), (2.18) and (2.20), we have

$$\nabla p = \alpha[e_+ I_{n-1}(kr) + e_- I_{n+1}(kr) + \mathrm{i}\sqrt{2}\, e_z I_n(kr)]\, e^{\mathrm{i}n\varphi}\, e^{\mathrm{i}kr} \tag{2.34}$$

where α is a constant (proportional to k); inserting (2.33), (2.34) and (2.13) into (2.4), we obtain

$$\begin{aligned} &e_+[(D^-_n D^+_{n-1} + \lambda^2 - k^2) W_-(r) + \alpha I_{n-1}(kr)] \\ &\qquad + e_-[(D^+_n D^-_{n+1} + \lambda^2 - k^2) W_+(r) + \alpha I_{n+1}(kr)] \\ &\qquad + e_z[(D^-_{n+1} D^+_n + \lambda^2 - k^2) W_z(r) + \mathrm{i}\sqrt{2}\, \alpha I_n(kr)] = 0. \end{aligned}$$

The vectors e_+, e_-, e_z are independent, therefore each square bracket must equal zero separately. As $(D^-_{n+1}D^+_n - k^2) I_n(kr) = 0$, a particular solution of the last of these equations is $W_z = (-\mathrm{i}\alpha\sqrt{2}/\lambda^2) I_n(kr)$; the general solution finite at $r = 0$ is

$$W_z(r) = \gamma_z J_n(ar) - \frac{\mathrm{i}\alpha\sqrt{2}}{\lambda^2} I_n(kr) \tag{2.35}$$

where γ_z is an arbitrary constant and

$$a = \sqrt{(\lambda^2 - k^2)}. \tag{2.36}$$

In the same way we find

$$W_\pm(r) = \gamma_\pm J_{n\pm1}(ar) - \frac{\alpha}{\lambda^2} I_{n\pm1}(kr). \tag{2.37}$$

The values of α, $\gamma_\pm$, γ_z and a are determined by the boundary conditions

$$\gamma_\pm J_{n\pm1}(aR) = \frac{\alpha}{\lambda^2} I_{n\pm1}(kR) \tag{2.38}$$

$$\gamma_z J_n(aR) = \frac{\mathrm{i}\alpha\sqrt{2}}{\lambda^2} I_n(kR) \tag{2.39}$$

and the incompressibility condition

$$\begin{aligned}\nabla \cdot w &= \left(\partial_r + \frac{1}{r}\right) w_r + \frac{1}{r}\partial_\varphi w_\varphi + \partial_z w_z \\ &= [2^{-1/2}(D^-_{n+1} W_+ + D^+_{n-1} W_-) + \mathrm{i}kW_z] \quad \mathrm{e}^{\mathrm{i}n\varphi}\,\mathrm{e}^{\mathrm{i}kz} = 0.\end{aligned} \tag{2.40}$$

Inserting (2.35) and (2.37) into (2.40), we get

$$\gamma_z = \frac{\mathrm{i}a}{k\sqrt{2}}(\gamma_+ - \gamma_-). \tag{2.41}$$

Suppose that $\alpha = 0$; then, to satisfy (2.38), (2.39) and (2.41), either a or two of the three constants γ_+, γ_-, γ_z must also be zero. It can easily be shown that both cases lead to the trivial solution $w = 0$. Therefore $\alpha \neq 0$ and we may put $\alpha = \lambda^2$. Expressing $\gamma_\pm$ and γ_z from (2.38) and (2.39) and inserting into (2.41), we find that a must be a non-zero solution of the equation

$$\frac{2k}{a}\frac{I_n(kR)}{J_n(aR)} = \frac{I_{n+1}(kR)}{J_{n+1}(aR)} - \frac{I_{n-1}(kR)}{J_{n-1}(aR)}. \tag{2.42}$$

If a is a root of (2.42), then so is $-a$; however, the two roots lead to the same eigenvalue and eigenfunction. We can easily prove that (2.42) has no imaginary roots: let a be a solution of (2.42); then $-a^2$ is an eigenvalue of the self-adjoint operator $\tilde{\Delta} - \partial_z^2$. If $v \in \mathscr{F}$, then $\tilde{\Delta} v = \Delta v + \nabla q$ for some scalar q and

$$\begin{aligned}\int v^* \cdot (\tilde{\Delta} - \partial_z^2) v \,\mathrm{d}V &= \int v^* \cdot (\Delta - \partial_z^2) v \,\mathrm{d}V \\ &= -\int (|\partial_x v|^2 + |\partial_y v|^2)\,\mathrm{d}V.\end{aligned}$$

Therefore $\tilde{\Delta} - \partial_z^2$ is negative definite, $-a^2$ must be real and negative and a is real. We see that it is enough to consider real positive values of a. We come to the conclusion that for $k > 0$, $n > 0$ the eigenfunctions $w = w_{knl}$ are given by

$$\begin{aligned} w_{knl}(r, \varphi, z) = N_{knl}\bigg[&\left(\frac{I_{n+1}(kR)}{J_{n+1}(aR)} J_{n+1}(ar) - I_{n+1}(kr)\right) e_- \\ &+ \left(\frac{I_{n-1}(kR)}{J_{n-1}(aR)} J_{n-1}(ar) - I_{n-1}(kr)\right) e_+ \\ &+ \mathrm{i}\sqrt{2}\left(\frac{I_n(kR)}{J_n(aR)} J_n(ar) - I_n(kr)\right) e_z\bigg]\,\mathrm{e}^{\mathrm{i}n\varphi}\,\mathrm{e}^{\mathrm{i}kz}\end{aligned} \tag{2.43}$$

where $a \equiv a_{knl}$ is the lth positive root of (2.42); the corresponding eigenvalue is

$$-\lambda^2_{knl} = -a^2_{knl} - k^2. \tag{2.44}$$

Case 2. $k>0$, $n=0$. Up to (2.41) the reasoning is the same as in the previous case. Then we use the fact that $J_1(x)=-J_1(x)$ and $I_{-1}(x)=I_1(x)$. This means, first, that (2.42) can be written as

$$L(a)\equiv\frac{J_1(aR)}{aRJ_0(aR)}-\frac{I_1(kR)}{kRI_0(kR)}=0 \tag{2.45}$$

second, that the coefficients at $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ are equal in (2.43), and third, that the system (2.38), (2.39), (2.41) admits yet another solution: $\alpha=\gamma_z=0$, $\gamma_+=\gamma_-$, when a is a positive root of

$$J_1(aR)=0. \tag{2.46}$$

The second term in (2.45) is a constant (independent of a) whose value lies between 0 and $\frac{1}{2}$; the qualitative behaviour of the first term, $J_1(aR)/aRJ_0(aR)$, is seen in figure 2. We see that the roots l_1, $l_2,\ldots$ of (2.45) and the roots j_2, $j_2,\ldots$ of (2.46) interweave in the order j_1, l_1, j_2, $l_2,\ldots$. Therefore the eigenfunctions in case 2 are

$$\boldsymbol{w}_{k0l}=N_{k0l}(\boldsymbol{e}_+-\boldsymbol{e}_-)J_1(ar)\,\mathrm{e}^{\mathrm{i}kz} \qquad \text{for } l \text{ odd} \tag{2.47}$$

$$\boldsymbol{w}_{k0l}=N_{k0l}\left[\left(\frac{I_1(kR)}{J_1(aR)}J_1(ar)-I_1(kr)\right)(\boldsymbol{e}_++\boldsymbol{e}_-)+\mathrm{i}\sqrt{2}\left(\frac{I_0(kR)}{J_0(aR)}J_0(ar)-I_0(kr)\right)\boldsymbol{e}_z\right]\mathrm{e}^{\mathrm{i}kz} \qquad \text{for } l \text{ even} \tag{2.48}$$

where $a=a_{k0l}$ is the lth positive root of

$$J_1(aR)\left(\frac{J_1(aR)}{aJ_0(aR)}-\frac{I_1(kR)}{kI_0(kR)}\right)=0. \tag{2.49}$$

Note that $\boldsymbol{e}_+-\boldsymbol{e}_-=\mathrm{i}\sqrt{2}\,\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_++\boldsymbol{e}_-=\sqrt{2}\,\boldsymbol{e}_r$; therefore $\boldsymbol{w}_{k0l}$ is polarised in the direction of $\boldsymbol{e}_\varphi$ for l odd and in the $(\boldsymbol{e}_r, \boldsymbol{e}_z)$ plane for l even.

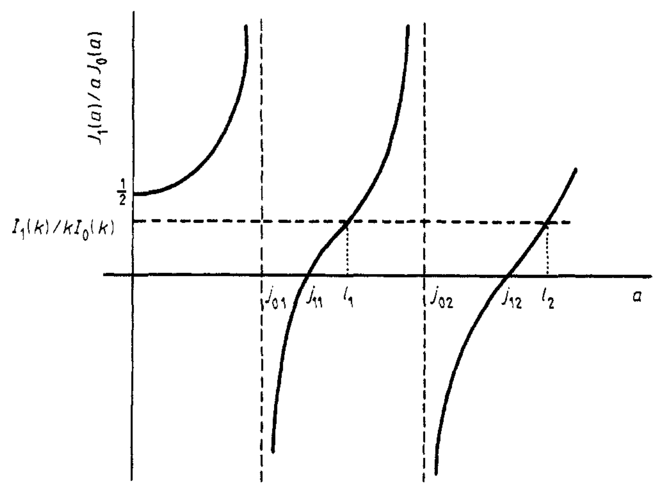

Figure 2. Qualitative behaviour of $J_1(a)/aJ_0(a)$.

Case 3. $k=0$, $n>0$. Now, the pressure gradient is, according to (2.26), (2.17), and (2.25),

$$\nabla p = \alpha e_+ r^{n-1} \mathrm{e}^{\mathrm{i}n\varphi}. \tag{2.50}$$

We insert this and (2.33) into (2.4):

$$e_+[(D_n^- D_{n-1}^+ + \lambda^2) W_-(r) + \alpha r^{n-1}] + e_-(D_n^+ D_{n+1}^- + \lambda^2) W_+(r) + e_z(D_{n+1}^- D_n^+ + \lambda^2) W_z = 0. \tag{2.51}$$

The general solution of this system is

$$W_-(r) = -\frac{\alpha}{\lambda^2} r^{n-1} + \gamma_- J_{n-1}(ar) \tag{2.52}$$

$$W_+(r) = \gamma_+ J_{n+1}(ar) \tag{2.53}$$

$$W_z(r) = \gamma_z J_n(ar) \tag{2.54}$$

where $a=\lambda$. The incompressibility condition (2.40) implies

$$(\gamma_+ - \gamma_-) J_n(ar) = 0 \tag{2.55}$$

for all r, or

$$\gamma_+ = \gamma_-. \tag{2.56}$$

The boundary conditions are

$$\gamma_+ J_{n-1}(aR) = \frac{\alpha}{\lambda^2} R^{n-1} \tag{2.57}$$

$$\gamma_- J_{n+1}(aR) = 0 \tag{2.58}$$

$$\gamma_z J_n(aR) = 0. \tag{2.59}$$

We add (2.57) and (2.58), taking into account (2.56) and the well known relation $J_{n-1}(aR) + J_{n+1}(aR) = (2n/aR) J_n(aR)$:

$$\gamma_+ J_n(aR) = \frac{\alpha}{2na} R^n \tag{2.60}$$

(recall that $\lambda = a$). One solution of the system (2.56)–(2.59) is $\alpha = \gamma_+ = \gamma_- = 0$, $\gamma_z \neq 0$, $J_n(aR) = 0$, the other is $\gamma_z = 0$, $J_{n+1}(aR) = 0$ and

$$\gamma_+ = \gamma_- = \frac{\alpha R^n}{2na J_n(aR)} \neq 0. \tag{2.61}$$

Since the zeros of the functions J_n and J_{n+1} follow in the order $j_{n,1}, j_{n+1,1}, j_{n,2}, j_{n+1,2}, \ldots$, the eigenfunctions are

$$w_{0nl} = N_{0nl}\, \mathrm{i} e_z J_n(ar)\, \mathrm{e}^{\mathrm{i}n\varphi} \qquad \text{for } l \text{ odd} \tag{2.62}$$

$$w_{0nl} = N_{0nl}\left[e_- \frac{R^{n-1}}{J_{n-1}(aR)} J_{n+1}(ar) + e_+\left(\frac{R^{n-1}}{J_{n-1}(aR)} J_{n-1}(ar) - r^{n-1} \right)\right] \mathrm{e}^{\mathrm{i}n\varphi} \qquad \text{for } l \text{ even} \tag{2.63}$$

where $a = a_{0nl}$ is the lth positive root of

$$J_n(aR)J_{n+1}(aR) = 0. \tag{2.64}$$

The eigenvalue is $-\lambda^2_{0nl} = -a^2_{0nl}$.

Case 4. $k=0$, $n=0$. The pressure gradient is zero by (2.17), and (2.51) (with $\alpha = 0$) is solved by

$$W_\pm(r) = \pm\gamma_\pm J_1(ar) \tag{2.65}$$

$$W_z(r) = \gamma_z J_0(ar). \tag{2.66}$$

The boundary conditions are

$$\gamma_\pm J_1(aR) = 0 \tag{2.67}$$

$$\gamma_z J_0(aR) = 0 \tag{2.68}$$

and the incompressibility condition (2.55) again implies (2.56). The solution of (2.56) and (2.67), (2.68) is either $\gamma_+ = \gamma_- = 0$, $\gamma_z \neq 0$, $J_0(aR) = 0$, or $\gamma_z = 0$, $\gamma_+ = \gamma_- \neq 0$, $J_1(aR) = 0$. Therefore the eigenfunctions are

$$w_{00l} = \begin{cases} N_{00l}e_z\, \mathrm{i}J_0(ar) & \text{for } l \text{ odd} \quad (2.69)\\ N_{00l}(e_+ - e_-)J_1(ar) & \text{for } l \text{ even} \quad (2.70)\end{cases}$$

where $a = a_{00l}$ is the lth positive root of

$$J_0(aR)J_1(aR) = 0. \tag{2.71}$$

3. Discrete symmetries and the definition of w_{knl} for negative k or n

Let T_φ be the inversion against the plane $\varphi = 0$, $\varphi = \pi$, or the transformation $\varphi \to -\varphi$, $e_\varphi \to -e_\varphi$, $e_\pm \to e_\mp = e^*_\pm$, and let T_z be the inversion against the plane $z = 0$, or the transformation $z \to -z$, $e_z \to -e_z$. Evidently, these operators commute with $\tilde{\Delta}$ and with each other, T_z anticommutes with ∂_z, and T_φ with ∂_φ; the boundary and incompressibility conditions are preserved by T_φ and T_z. This means that if w is an eigenfunction of $\tilde{\Delta}$, $-\mathrm{i}\partial_z$ and j with the eigenvalues $-\lambda^2$, k, n, then $T_z w$ and $T_\varphi w$ are eigenfunctions of the same three operators with the eigenvalues, respectively, $-\lambda^2$, $-k$, n and $-\lambda^2$, k, $-n$. The eigenvalues $-\lambda^2_{knl}$ with $k = n = 0$ are thus fourfold degenerate, those with $k = 0$, $n \neq 0$ or $k \neq 0$, $n = 0$ are twofold degenerate, and the rest are non-degenerate.

Though the eigenfunctions with negative values of k or n can be found in the same way as those with positive k and n, it is convenient to define them by

$$w_{-knl} = T_z w_{knl} \tag{3.1}$$

$$w_{k-nl} = T_\varphi w_{knl} \tag{3.2}$$

as this establishes a definite phase. (The phase of w_{knl} with k, $n \geq 0$ is fixed by the requirement that $N_{knl} > 0$.) With this choice of phase, we always have $W_\pm$ real and W_z purely imaginary. Since the operation of $T_z T_\varphi$ consists of exchanging $e_+ \to e_- = e^*_+$ and vice versa, $e_z \to -e_z$ and $\mathrm{e}^{\mathrm{i}n\varphi}\,\mathrm{e}^{\mathrm{i}kz} \to \mathrm{e}^{-\mathrm{i}n\varphi}\,\mathrm{e}^{-\mathrm{i}kz}$, all the eigenfunctions become complex conjugate on the simultaneous inversion:

$$T_z T_\varphi w_{knl} = w^*_{knl}. \tag{3.3}$$

If $k=0$, then w_{knl} is an eigenfunction of T_z,

$$T_z w_{0nl} = \beta^z_{0nl} w_{0nl}. \tag{3.4}$$

Since $T_z^2 = 1$, $\beta^z_{0nl} = \pm 1$; and the same is true for the parity with respect to T_φ,

$$T_\varphi w_{k0l} = \beta^\varphi_{k0l} w_{k0l}. \tag{3.5}$$

Inspecting the expressions for the eigenfunctions, we find that

$$\beta^\varphi_{00l} = (-1)^{l+1} \tag{3.6}$$

$$\beta^\varphi_{k0l} = (-1)^l \qquad \text{for } k \neq 0 \tag{3.7}$$

$$\beta^z_{0nl} = (-1)^l \qquad \text{for all } n. \tag{3.8}$$

From (3.3) we see that w^*_{knl} is equal to w_{-k-nl}, with a possible change of sign. Therefore the basis contains with each function also its complex conjugate (up to the sign).

Note that as $k \to 0$, the transition of w_{knl} to w_{0nl} is continuous if $n \neq 0$ and discontinuous if $n = 0$. Indeed, as $I_n(x) \sim (x/2)^n/n!$ for $n > 0$ and small x, equation (2.42) divided by $(kR)^{n-1}/2^{n+1}(n+1)!$ gives

$$\frac{4(n+1)(kR)^2}{aRJ_n(aR)} \sim \frac{(kR)^2}{J_{n+1}(aR)} - \frac{4n(n+1)}{J_{n-1}(ar)} \tag{3.9}$$

to the lowest order in kR, or

$$J_n(aR)J_{n+1}(aR) \sim \text{constant} \times (kR)^2. \tag{3.10}$$

This equation approaches (2.64) or (2.71) as $k \to 0$; we express either $J_n(aR)$ or $J_{n+1}(aR)$ as constant $\times\,(kR)^2$, insert into (2.43) or (2.47), (2.48) and find that

$$\lim_{k \to 0} w_{knl} = w_{0nl} \qquad n \neq 0. \tag{3.11}$$

However, if $n = 0$, this argument fails, because $I_{-1}(x) \sim (x/2)^1$; and indeed, from (2.47), (2.48) and (2.69), (2.70) we can see that only the eigenvalues and eigenfunctions with odd l are continuous functions of k at $k = 0$. When l is even, the roots a_{k0l} of (2.49) tend to the positive zeros of $J_2(aR)$, and not to the corresponding roots of (2.71); this explains the exchange of parity of w_{k0l} as k becomes zero (see (3.6), (3.7)). This somewhat counter-intuitive result is explained as follows. Equations (2.38), (2.39) and (2.41) are four homogeneous linear equations for four unknown variables γ_+, γ_-, γ_z, α, and we seek their non-trivial solutions; equation (2.42) and its analogues are actually conditions for the determinant of the system to be zero. From (2.26) and (2.20) we see that α is proportional to k; rescaling the variable α, we can get a determinant which is not proportional to k if $n \neq 0$, while no such rescaling is possible if $n = 0$. Therefore, as k becomes zero, the rank of the matrix is decreased by one and this causes the jump in the eigenvalues and eigenfunctions.

4. Normalization

To find the normalization factors N_{knl}, we use the standard method of calculating norms of eigenfunctions of self-adjoint operators. Let S be a formally (i.e. without domain) defined operator, let a subspace $\mathcal{F}$ be defined by one or more linear homogeneous equations $Hw = 0$, and S be self-adjoint on $\mathscr{F}$. When solving the eigenproblem of S on $\mathscr{F}$, we first look for all the solutions of the equation

$$Sw_a = \mu_a w_a \tag{4.1}$$

where a denotes one or more parameters which label the eigenfunctions and eigenvalues. Then we insert the general solution of (4.1) into the equation $Hw = 0$ and seek the set A of all the values of a for which

$$Hw_a = 0 \tag{4.2}$$

has a non-trivial solution. Let $a \in A$, $b \notin A$; then

$$(\mu_b - \mu_a) \int w_b^* \cdot w_a \, \mathrm{d}V = \int (Sw_b^* \cdot w_a - w_b^* \cdot Sw_a) \, \mathrm{d}V. \tag{4.3}$$

The right-hand side of (4.3) is non-zero, but becomes zero as $b \to a$, since S is self-adjoint on $\mathscr{F}$. Therefore the normalization integral is

$$\int |w_a|^2 \, \mathrm{d}V = \left(\frac{\mathrm{d}\mu_b}{\mathrm{d}b}\right)^{-1} \frac{\mathrm{d}}{\mathrm{d}b} \int (Sw_b^* \cdot w_a - w_b^* \cdot Sw_a) \, \mathrm{d}V \bigg|_{b=a}. \tag{4.4}$$

As a rule, the right-hand side reduces to a surface integral, which can, in systems like the cylindrical pipe, be easily evaluated.

Now we apply this general procedure to our particular system. We fix the values of k and n and consider the equation

$$\tilde{\Delta} w_a + (a^2 + k^2) w_a = 0 \qquad \text{or} \qquad \Delta w_a + \lambda_a^2 w_a + \nabla(\lambda_a^2 q) = 0 \tag{4.5}$$

where $\lambda_a^2 = a^2 + k^2$ and q is a scalar function. So far w_a does not satisfy the incompressibility and boundary conditions; however, we desire that, as the coefficients $\gamma_\pm$ and γ_z and the parameter a assume the values required for fulfilling these conditions, the function w_a becomes the unnormalized eigenfunction w_{knl}/N_{knl} defined in section 2. Therefore we choose, for $k > 0$, $n > 0$,

$$q = \frac{\sqrt{2}}{k} I_n(kr) \, \mathrm{e}^{in\varphi} \, \mathrm{e}^{ikz} \equiv Q(r) \, \mathrm{e}^{in\varphi} \, \mathrm{e}^{ikz} \tag{4.6}$$

so that

$$\nabla q = (e_+ I_{n-1}(kr) + e_- I_{n+1}(kr) + i\sqrt{2} e_z I_n(kr)) \, \mathrm{e}^{in\varphi} \, \mathrm{e}^{ikz}$$

(comparing with (2.34) we see that $p = \alpha q = \lambda^2 q$). By (4.5) and the Gauss theorem we have

$$\begin{aligned}
(\lambda_b^2 - \lambda_a^2) & \int w_a^* \cdot w_b \, \mathrm{d}V \\
&= \int [(\Delta w_a + \lambda_a^2 \nabla q)^* \cdot w_b - w_a^* \cdot (\Delta w_b + \lambda_b^2 \nabla q)] \, \mathrm{d}V \\
&= \int_C [(\partial_n w_a^*) \cdot w_b - w_a^* \cdot \partial_n w_b + \lambda_a^2 q^* w_{bn} - \lambda_b^2 q w_{an}] \, \mathrm{d}S \\
&\quad + \int (\lambda_b^2 q \nabla \cdot w_a - \lambda_a^2 q^* \nabla \cdot w_b) \, \mathrm{d}V
\end{aligned} \tag{4.7}$$

where C is the cylinder $r=R$, $0\leqslant z\leqslant Z$, ∂_n denotes normal derivatives, $\partial_n=\partial_r$, and w_{an} etc denote normal components, e.g. $w_{an}=2^{-1/2}(w_{a+}+w_{a-})$.

Now, in the expression for $\boldsymbol{w}_a$,

$$\boldsymbol{w}_a=[(\gamma_{a+}J_{n+1}(ar)-I_{n+1}(kr))\boldsymbol{e}_-+(\gamma_{a-}J_{n-1}(ar)-I_{n-1}(kr))\boldsymbol{e}_+ \\ +\mathrm{i}\sqrt{2}(\gamma_{az}J_n(ar)-I_n(kr))\boldsymbol{e}_z]\,\mathrm{e}^{\mathrm{i}n\varphi}\,\mathrm{e}^{\mathrm{i}kz} \tag{4.8}$$

we put

$$\gamma_{a\pm}=\frac{I_{n\pm1}(kR)}{J_{n\pm1}(aR)}\qquad \gamma_{az}=\frac{I_n(kR)}{J_n(aR)} \tag{4.9}$$

and for a we choose a positive root of (2.42). This implies that $\boldsymbol{w}_a\in\mathscr{F}$:

$$\boldsymbol{w}_a=0\quad\text{on } C\qquad \nabla\cdot\boldsymbol{w}_a=0. \tag{4.10}$$

In the analogous expression for $\boldsymbol{w}_b$ we put

$$\gamma_{b\pm}=\gamma_{a\pm}\qquad \gamma_{bz}=\frac{b}{2k}(\gamma_{b+}-\gamma_{b-}).$$

Then we have $\nabla\cdot\boldsymbol{w}_b=0$, but $\boldsymbol{w}_b\neq0$ on C, and (4.7) simplifies to

$$(\lambda_b^2-\lambda_a^2)\int \boldsymbol{w}_a^*\cdot\boldsymbol{w}_b\,\mathrm{d}V=\int_C(\partial_n\boldsymbol{w}_a^*\cdot\boldsymbol{w}_b+\lambda_a^2q^*w_{bn})\,\mathrm{d}S. \tag{4.11}$$

We use the fact that $\lambda_b^2-\lambda_a^2=b^2-a^2$, differentiate with respect to b and put $b=a$:

$$\int|\boldsymbol{w}_a|^2\,\mathrm{d}V=\frac{1}{2a}\int_C\left(\partial_n\boldsymbol{w}_a^*\cdot\frac{\partial\boldsymbol{w}_b}{\partial b}\bigg|_{b=a}+\lambda_a^2q^*\frac{\partial w_{bn}}{\partial b}\bigg|_{b=a}\right)\mathrm{d}S \\ =\frac{\pi RZ}{a}\left(\partial_r\boldsymbol{W}_a^*\cdot\frac{\partial\boldsymbol{W}_b}{\partial b}\bigg|_{b=a}+\lambda_a^2Q^*\frac{\partial W_{br}}{\partial b}\bigg|_{b=a}\right). \tag{4.12}$$

Since we have, e.g.,

$$W_{b+}=\frac{I_{n+1}(kR)}{J_{n+1}(aR)}J_{n+1}(br)-I_{n+1}(kr)$$

the differentiation affects only $J_{n+1}(br)$ etc. After some manipulation, (4.12) yields

$$N_{knl}^{-2}=\int|\boldsymbol{w}_a|^2\,\mathrm{d}V \\ =\pi R^2Z\bigg[\left(\gamma_-J'_{n-1}-\frac{k}{a}I'_{n-1}\right)\gamma_-J'_{n-1}+\left(\gamma_+J'_{n+1}-\frac{k}{a}I'_{n+1}\right)\gamma_+J'_{n+1} \\ +2\left(\gamma_zJ'_n-\frac{k}{a}I'_n\right)\left(\gamma_zJ'_n+\frac{1}{aR}I_n\right) \\ +\frac{a^2+k^2}{ak}(\gamma_+J'_{n+1}+\gamma_-J'_{n-1})I_n\bigg]\qquad k\neq0,\ n\neq0 \tag{4.13}$$

where the arguments of J and J' are $a_{knl}R$, those of I and I' are kR, and γ_+, γ_-, γ_z are given by (4.9) with $a=a_{knl}$.

This result is derived for $k\neq 0$, $m\neq 0$ and, as can be seen from section 2, remains valid also for $k\neq 0$, $n=0$, l even. However, in the latter case the expression (4.13) can be considerably simplified, using the identities $\gamma_+=-\gamma_-=(k/a)\gamma_z$, $J_0'=-J_1$, $I_0'=I_1$:

$$N_{k0l}^{-2}=2\pi R^2 Z\left[\left(1+\frac{a^2}{k^2}\right)I_1^2+\left(1+\frac{k^2}{a^2}\right)I_0 I_2\right]\qquad k\neq 0,\ l \text{ even.} \tag{4.14}$$

For $k\neq 0$, $n=0$, l odd, the normalization integral can be easily deduced from (2.47) and the well known expression for $\int_0^1 J_1(ar)^2 r\,\mathrm{d}r$ when $J_1(a)=0$:

$$N_{k0l}^{-2}=2\pi R^2 Z J_0(aR)^2\qquad k\neq 0,\ l \text{ odd.} \tag{4.15}$$

Normalization integrals can be calculated equally simply in some other cases, as seen from (2.62), (2.69) and (2.70):

$$N_{0ml}^{-2}=\pi R^2 Z J_{n+1}(aR)^2\qquad l \text{ odd} \tag{4.16}$$

$$N_{00l}^{-2}=2\pi R^2 Z J_0(aR)^2\qquad l \text{ even.} \tag{4.17}$$

The remaining case $k=0$, $n\neq 0$, l even, can be treated in the same manner as the case $k\neq 0$, $n\neq 0$, which we have described above. The result, after all possible simplifications, is

$$N_{0nl}^{-2}=\frac{\pi Z a^2 R^{2n+2}}{2n^2}\qquad n\neq 0,\ l \text{ even.} \tag{4.18}$$

We see that except for (4.13) the expressions for N_{knl}^{-2} are compact and manifestly positive. I have not been successful in rewriting the right-hand side of (4.13) to an analogous form, though it seems probable that such a form exists.

5. Generalization

In the introduction we have mentioned the convenience of writing the Navier-Stokes equations in the basis of solenoidal vector functions satisfying the boundary conditions. We have found explicit expressions for the basis functions in the cylindrical geometry; owing to the high symmetry and well known properties of the cylindrical functions this was a relatively easy task. Of course, we often need to study the Navier-Stokes equations in other geometries; how can we construct the basis there?

To see this, we extract what is general from the procedure described in section 2. First we find the general expression for pressure p: as mentioned at (2.15), p must be a harmonic function. We find its gradient and insert it into (2.4):

$$(\Delta+\lambda^2)\boldsymbol{w}+\alpha\nabla p=0. \tag{5.1}$$

Let $\boldsymbol{v}$ be a general solution of the vector eigenvalue equation

$$(\Delta+\lambda^2)\boldsymbol{v}=0$$

(if we are using constant unit vectors, such as $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$, this equation reduces to three scalar eigenvalue equations of the same familiar type). Then the general solution of (5.1) is

$$\boldsymbol{w}=\boldsymbol{v}-\frac{\alpha}{\lambda^2}\nabla p. \tag{5.3}$$

We insert this expression into the boundary and incompressibility conditions and fix the eigenvalue by requiring that the resulting linear homogeneous equations for α and the free coefficients γ_i of $\boldsymbol{v}$ have a non-trivial solution. Finally we find this solution and calculate the normalization integral by the method described in section 4. In some of the most frequently studied problems, such as plane Poisseuille flow, we can use as many symmetries as in the pipe flow problem, and the construction of the basis is then equally easy. Note that in the general procedure just described we solve well studied equations—the Laplace equation for p and (5.2) for $\boldsymbol{v}$; in many geometries the solution of these equations is already known.

So far we were assuming that the boundary conditions are homogeneous, and consequently, that $\mathcal{F}$ is a linear space. If the condition $\boldsymbol{v}=0$ on the walls is not satisfied (moving walls or flow through boundary), then $\mathcal{F}$ is a linear manifold and not a linear space. However, shifting $\mathcal{F}$ by a constant vector, we have a space again. We subtract from $\boldsymbol{v}(\boldsymbol{x}, t)$ a constant vector field $\boldsymbol{v}_0(x)$ satisfying the incompressibility and boundary conditions, and expand the rest in a basis found as described above:

$$\boldsymbol{v}(x, t) = \boldsymbol{v}_0(x) + \sum v_I(t)\boldsymbol{w}_I(x).$$

References

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